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A CONTRIBUTION TO THE QUESTION OF LINEAR DEPENDENCE
IN LINEAR INTEGRAL EQUATIONS.



A

DISSERTATION

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A Contribution to the Question of Linear Dependence in Linear Integral Equations,

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A set of m linear algebraic functions in n variables

$$(a) \quad \sum_{j=1}^n a_{ij} x_j = y_i, \quad i=1, 2, 3, \dots, m$$

is said to be linearly dependent if there exists a set of m constants c_i not all zero such that

$$\sum_{i=1}^m c_i y_i \equiv 0.$$

If each function y_i of (a) be multiplied through by the corresponding c_i and the sum of the products taken, the above definition is easily seen to be equivalent to

Theorem I. A necessary and sufficient condition that the set (a) be linearly dependent is that there exists a set of m constants c_i not all zero such that

$$(b) \quad \sum_{i=1}^m c_i a_{ij} = 0, \quad j=1, 2, 3, \dots, n.$$

Again if (b) be regarded as a homogeneous system of n equations in the c_i , we know it can have a non-zero solution, $c_1, c_2, \dots, c_m$, if and only if the rank of the matrix

$$|| a_{ij} || \quad i=1, 2, 3, \dots, m; \quad j=1, 2, 3, \dots, n$$

is less than m . Hence our definition of linear dependence is also equivalent to

Theorem II. A necessary and sufficient condition that the set (a) be linearly dependent is that the rank of the system of m equations

$$(c) \quad \sum_{j=1}^n a_{ij} x_j = 0, \quad i=1, 2, 3, \dots, m$$

be less than m .

Suppose $m < n$. Then from Theorem II it follows that if the func-

i.e. the rank of the first m equations of (e) must be less than m , and we thus have

Theorem III. A necessary and sufficient condition that the set (a) for $m < n$ shall be linearly dependent is that for any arbitrary set of a_{nj} , the system of equations (e) have a non-zero solution $x_1, x_2, \dots, x_n$, i.e. the determinant of the set (d) be identically zero in the a_{nj} .

We propose in this paper to consider linear dependence relative to linear integral expressions involving two or more functions $\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)$, i.e. expressions of the type

$$\varphi_i(x) - \sum_{j=1}^n \int_a^b k_{ij}(x, y) \varphi_j(y) dy = f_i(x), \quad i=1, 2, 3, \dots, m.$$

We shall give as our definition of linear dependence for a set of such linear integral expressions the analogue to Theorem III above, viz:

Definition. A set of m linear integral expressions

$$(1) \quad \varphi_i(x) - \sum_{j=1}^n \int_a^b k_{ij}(x, y) \varphi_j(y) dy = f_i(x), \quad i=1, 2, 3, \dots, m$$

is said to be linearly dependent if n functions $\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)$, not all zero, exist which satisfy the system of equations

$$(2) \quad \begin{aligned} \varphi_i(x) - \sum_{j=1}^n \int_a^b k_{ij}(x, y) \varphi_j(y) dy &= 0, & i=1, 2, \dots, m, \\ \varphi_h(x) - \sum_{j=1}^n \int_a^b k_{hj}(x, y) \varphi_j(y) dy &= 0, & h=m+1, m+2, \dots, n, \end{aligned}$$

for $k_{nj}(x, y)$ entirely arbitrary, i.e. if the Fredholm determinant of (2) is identically zero for all $k_{nj}(x, y)$.

In Part I we study the consequences of this definition with respect to a single linear integral expression in two functions

$$(3) \quad \varphi_1(x) - \int_a^b k_{11}(x, y) \varphi_1(y) dy - \int_a^b k_{12}(x, y) \varphi_2(y) dy = f_1(x),$$

in which, for convenience, $\varphi_1(x), \varphi_2(x)$ and $f_1(x)$ are assumed to be continuous on the interval $a \leq x \leq b$ and $k_{11}(x, y), k_{12}(x, y)$, continuous in the square $a \leq x \leq b, a \leq y \leq b$ (1).

(1) Throughout this paper we shall limit ourselves to functions $\varphi(x)$ and $f(x)$ continuous in $a \leq x \leq b$ and functions $k(x, y)$ continuous in the square $a \leq x \leq b, a \leq y \leq b$. For other possible conditions cf. Bôcher, An Introduction to the Study of Integral Equation, p. 3. We shall also assume that our integrations are extended over the interval $a \leq x \leq b$.

We develop first in § 1, some interesting expansions of Fredholm determinants, of certain combinations of kernels from which we deduce, in § 2, some necessary and sufficient conditions for linear dependence relative to equation (3). In § 3 we establish for (3) a theorem analogous to Theorem I above.

In Part II we consider a set of m linear integral expressions in n functions

$$(4) \quad \varphi_i(x) - \sum_{j=1}^n \int k_{ij}(x, y) \varphi_j(y) dy = f_i(x) \quad i=1, 2, 3, \dots, m,$$

to the form of which the more general set

$$(4') \quad \sum_{j=1}^n a_{ij}(x) \varphi_j(x) - \sum_{j=1}^n \int k_{ij}(x, y) \varphi_j(y) dy = f_i(x) \quad i=1, 2, 3, \dots, m$$

can always be reduced provided there exists an m -rowed determinant

$$A(x) = |a_{il}(x)| \quad i, l=1, 2, 3, \dots, m$$

which does not vanish on the interval $a \leq x \leq b$ ⁽¹⁾.

Since the determinantal formulae developed in Part I become very complicated in the general case, we give only a treatment of this case analogous to that of Part I, § 3.

(¹) If we suppose for convenience that the determinant $|a_{il}(x)|$, ($i, l=1, 2, \dots, m$) does not vanish on the interval $a \leq x \leq b$, then by solving the equations (4') for $\varphi_1(x), \dots, \varphi_m(x)$, we get a system of the form

$$\varphi_i(x) + \sum_{h=m+1}^n b_{ih}(x) \varphi_h(x) - \sum_{l=1}^m \int \bar{k}_{il}(x, y) \varphi_l(y) dy - \sum_{h=m+1}^n \int \bar{k}_{ih}(x, y) \varphi_h(y) dy = \bar{f}_i(x) \quad i=1, 2, 3, \dots, m.$$

If now we set

$$\begin{aligned} \varphi_i(x) + \sum_{h=m+1}^n b_{ih}(x) \varphi_h(x) &= \Phi_i(x), \quad i=1, 2, 3, \dots, m \\ \varphi_h(x) &= \Phi_h(x), \quad h=m+1, m+2, \dots, n. \end{aligned}$$

Then these can be reduced to

$$\Phi_i(x) - \sum_{j=1}^n \int \bar{k}_{ij}(x, y) \Phi_j(y) dy = \bar{f}_i(x), \quad i=1, 2, 3, \dots, m,$$

where the $\bar{k}_{ij}(x, y)$ are easily expressible in terms of the $\bar{k}_{ij}(x, y)$ and so in terms of the $k_{ij}(x, y)$.

PART I.

§1. Fredholm⁽¹⁾ has suggested a method for finding the Fredholm determinant for a system of n integral equations

$$\varphi_g(x) - \sum_{j=1}^n \int k_{gj}(x, y) \varphi_j(y) dy = f_g(x), \quad g=1, 2, 3, \dots, n$$

which is equivalent to the formula

$$(1) \quad F_o[k_{gj}] = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \sum_{g_1=1}^n \int \sum_{g_2=1}^n \dots \dots \dots \sum_{g_m=1}^n \int K \begin{matrix} g_1 g_2 \dots g_m \\ g_1 g_2 \dots g_m \end{matrix} (x_1 x_2 \dots x_m) dx_1 dx_2 \dots dx_m,$$

where

$$K \begin{matrix} g_1 g_2 \dots g_m \\ j_1 j_2 \dots j_m \end{matrix} (x_1 x_2 \dots x_m) = \begin{vmatrix} k_{g_1 j_1}(x_1 y_1) & k_{g_1 j_2}(x_1 y_2) & \dots & k_{g_1 j_m}(x_1 y_m) \\ k_{g_2 j_1}(x_2 y_1) & k_{g_2 j_2}(x_2 y_2) & \dots & k_{g_2 j_m}(x_2 y_m) \\ \dots & \dots & \dots & \dots \\ k_{g_m j_1}(x_m y_1) & k_{g_m j_2}(x_m y_2) & \dots & k_{g_m j_m}(x_m y_m) \end{vmatrix}$$

In the case of the system of two equations

$$(2) \quad \begin{aligned} \varphi_1(x) - \int k_{11}(x, y) \varphi_1(y) dy - \int k_{12}(x, y) \varphi_2(y) dy &= f_1(x), \\ \varphi_2(x) - \int k_{21}(x, y) \varphi_1(y) dy - \int k_{22}(x, y) \varphi_2(y) dy &= f_2(x) \end{aligned}$$

(1) becomes

$$(3) \quad \begin{aligned} F_o \begin{bmatrix} k_{11} k_{12} \\ k_{21} k_{22} \end{bmatrix} &= 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \sum_{g_1=1}^2 \int \sum_{g_2=1}^2 \dots \dots \dots \sum_{g_m=1}^2 \int K \begin{matrix} g_1 g_2 \dots g_m \\ g_1 g_2 \dots g_m \end{matrix} (x_1 x_2 \dots x_m) dx_1 dx_2 \dots dx_m \\ &= 1 - \int [(k_{11}(x_1 x_1) + k_{22}(x_1 x_1)) dx_1 + \frac{1}{2!} \iint [K_{11} \begin{pmatrix} x_1 x_2 \\ x_1 x_2 \end{pmatrix} + 2 \begin{vmatrix} k_{11}(x_1 x_1) & k_{12}(x_1 x_2) \\ k_{21}(x_2 x_1) & k_{22}(x_2 x_2) \end{vmatrix} \\ &+ K_{22} \begin{pmatrix} x_1 x_2 \\ x_1 x_2 \end{pmatrix}] dx_1 dx_2 - \frac{1}{3!} \iiint [K_{11} \begin{pmatrix} x_1 x_2 x_3 \\ x_1 x_2 x_3 \end{pmatrix} \end{aligned}$$

(1) Acta Math. Vol. 27 (1903), p. 378. See also Platrier, Journal de Mathématiques, series 6, Vol. 9 (1913), p. 261.

$$\begin{aligned}
& + 3 \left| \begin{array}{ccc} k_{11}(x_1x_1) & k_{11}(x_1x_2) & k_{12}(x_1x_3) \\ k_{11}(x_2x_1) & k_{11}(x_2x_2) & k_{12}(x_2x_3) \\ k_{21}(x_3x_1) & k_{21}(x_3x_2) & k_{22}(x_3x_3) \end{array} \right| + 3 \left| \begin{array}{ccc} k_{11}(x_1x_1) & k_{12}(x_1x_2) & k_{12}(x_1x_3) \\ k_{21}(x_2x_1) & k_{22}(x_2x_2) & k_{22}(x_2x_3) \\ k_{21}(x_3x_1) & k_{22}(x_3x_2) & k_{22}(x_3x_3) \end{array} \right| \\
& + K_{22} \left(\frac{x_1x_2x_3}{x_1x_2x_3} \right) dx_1 dx_2 dx_3 + \frac{1}{4!} \iiint \left[K_{11} \left(\frac{x_1x_2x_3x_4}{x_1x_2x_3x_4} \right) \right. \\
& + 4 \left| \begin{array}{ccc} k_{11}(x_1x_1) & k_{11}(x_1x_2) & k_{11}(x_1x_3) & k_{12}(x_1x_4) \\ k_{11}(x_2x_1) & k_{11}(x_2x_2) & k_{11}(x_2x_3) & k_{12}(x_2x_4) \\ k_{11}(x_3x_1) & k_{11}(x_3x_2) & k_{11}(x_3x_3) & k_{12}(x_3x_4) \\ k_{21}(x_4x_1) & k_{21}(x_4x_2) & k_{21}(x_4x_3) & k_{22}(x_4x_4) \end{array} \right| \\
& + 6 \left| \begin{array}{ccc} k_{11}(x_1x_1) & k_{11}(x_1x_2) & k_{12}(x_1x_3) & k_{12}(x_1x_4) \\ k_{11}(x_2x_1) & k_{11}(x_2x_2) & k_{12}(x_2x_3) & k_{12}(x_2x_4) \\ k_{21}(x_3x_1) & k_{21}(x_3x_2) & k_{22}(x_3x_3) & k_{22}(x_3x_4) \\ k_{21}(x_4x_1) & k_{21}(x_4x_2) & k_{22}(x_4x_3) & k_{22}(x_4x_4) \end{array} \right| \\
& + 4 \left| \begin{array}{ccc} k_{11}(x_1x_1) & k_{12}(x_1x_2) & k_{12}(x_1x_3) & k_{12}(x_1x_4) \\ k_{21}(x_2x_1) & k_{22}(x_2x_2) & k_{22}(x_2x_3) & k_{22}(x_2x_4) \\ k_{21}(x_3x_1) & k_{22}(x_3x_2) & k_{22}(x_3x_3) & k_{22}(x_3x_4) \\ k_{21}(x_4x_1) & k_{22}(x_4x_2) & k_{22}(x_4x_3) & k_{22}(x_4x_4) \end{array} \right| \\
& \left. + K_{22} \left(\frac{x_1x_2x_3x_4}{x_1x_2x_3x_4} \right) dx_1 dx_2 dx_3 dx_4 - \dots \dots \dots \right]
\end{aligned}$$

By expanding still further according to the rows containing k_{11} and k_{12} and properly collecting terms, it is easily seen that after placing

$$x_{m+1}, x_{m+2}, \dots, x_{2m} = y_1, y_2, \dots, y_m \text{ and } x_{2m+1}, x_{2m+2}, \dots = s_1, s_2, \dots$$

(3) becomes

$$\begin{aligned}
(4) \quad F_{\circ} \left[\frac{k_{11} k_{12}}{k_{21} k_{22}} \right] &= F_{\circ}(k_{11}) F_{\circ}(k_{22}) + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \\
&\times \int \dots \int F_m \left(\frac{x_1 \dots x_m}{y_1 \dots y_m}; \frac{k_{12}}{k_{11}} \right) F_m \left(\frac{y_1 \dots y_m}{x_1 \dots x_m}; \frac{k_{21}}{k_{22}} \right) dx_1 \dots dx_m dy_1 \dots dy_m,
\end{aligned}$$

where

$$F_m \left(\frac{x_1 \dots x_m}{y_1 \dots y_m}; \frac{k_{12}}{k_{11}} \right) = K_{12} \left(\frac{x_1 \dots x_m}{y_1 \dots y_m} \right) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$$

$$\times \int \dots \int \begin{vmatrix} k_{12}(x_1 y_1) \dots k_{12}(x_1 y_m) & k_{11}(x_1 s_1) \dots k_{11}(x_1 s_n) \\ \dots & \dots \\ k_{12}(x_m y_1) \dots k_{12}(x_m y_m) & k_{11}(x_m s_1) \dots k_{11}(x_m s_n) \\ k_{12}(s_1 y_1) \dots k_{12}(s_1 y_m) & k_{11}(s_1 s_1) \dots k_{11}(s_1 s_n) \\ \dots & \dots \\ k_{12}(s_n y_1) \dots k_{12}(s_n y_m) & k_{11}(s_n s_1) \dots k_{11}(s_n s_n) \end{vmatrix} ds_1 ds_2 \dots ds_n$$

and $F_m \left(\begin{smallmatrix} y_1 \dots y_m \\ x_1 \dots x_m \end{smallmatrix}; \begin{smallmatrix} k_{21} \\ k_{22} \end{smallmatrix} \right)$ is similarly expressed.

If we expand this last expression according to minors of the first m columns and again collect terms properly, we obtain another expression for $F_m \left(\begin{smallmatrix} x_1 \dots x_m \\ y_1 \dots y_m \end{smallmatrix}; \begin{smallmatrix} k_{12} \\ k_{11} \end{smallmatrix} \right)$, viz :

$$\begin{aligned} F_m \left(\begin{smallmatrix} x_1 \dots x_m \\ y_1 \dots y_m \end{smallmatrix}; \begin{smallmatrix} k_{12} \\ k_{11} \end{smallmatrix} \right) &= F_o(k_{11}) K_{12} \left(\begin{smallmatrix} x_1 \dots x_m \\ y_1 \dots y_m \end{smallmatrix} \right) \\ &+ \int \sum_{i=1}^m F_1 \left(\begin{smallmatrix} x_i \\ s_1 \end{smallmatrix}; k_{11} \right) K_{12} \left(\begin{smallmatrix} x_1 \dots x_{i-1} s_1 x_{i+1} \dots x_m \\ y_1 \dots y_i \dots y_m \end{smallmatrix} \right) ds_1 + \dots \\ &+ \frac{1}{(q!)^2} \int \dots \int \sum_{i_1=1}^m \dots \sum_{i_q=1}^m F_q \left(\begin{smallmatrix} x_{i_1} \dots x_{i_q} \\ s_1 \dots s_q \end{smallmatrix}; k_{11} \right) \\ &\times K_{12} \left(\begin{smallmatrix} x_1 \dots x_{i_1-1} s_1 x_{i_1+1} \dots x_{i_q-1} s_q x_{i_q+1} \dots x_m \\ y_1 \dots y_{i_1} \dots y_{i_q} \dots y_m \end{smallmatrix} \right) ds_1 \dots ds_q + \dots \\ &+ \frac{1}{m!} \int \dots \int F_m \left(\begin{smallmatrix} x_1 \dots x_m \\ s_1 \dots s_m \end{smallmatrix}; k_{11} \right) K_{12} \left(\begin{smallmatrix} s_1 \dots s_m \\ y_1 \dots y_m \end{smallmatrix} \right) ds_1 \dots ds_m. \end{aligned}$$

In a similar way we obtain

$$\begin{aligned} F_m \left(\begin{smallmatrix} y_1 \dots y_m \\ x_1 \dots x_m \end{smallmatrix}; \begin{smallmatrix} k_{21} \\ k_{22} \end{smallmatrix} \right) &= F_o(k_{22}) K_{21} \left(\begin{smallmatrix} y_1 \dots y_m \\ x_1 \dots x_m \end{smallmatrix} \right) \\ &+ \int \sum_{i=1}^m F_1 \left(\begin{smallmatrix} y_i \\ s_1 \end{smallmatrix}; k_{22} \right) K_{21} \left(\begin{smallmatrix} y_1 \dots y_{i-1} s_1 y_{i+1} \dots y_m \\ x_1 \dots x_i \dots x_m \end{smallmatrix} \right) ds_1 + \dots \\ &+ \frac{1}{(q!)^2} \int \dots \int \sum_{i_1=1}^m \dots \sum_{i_q=1}^m F_q \left(\begin{smallmatrix} y_{i_1} \dots y_{i_q} \\ s_1 \dots s_q \end{smallmatrix}; k_{22} \right) \\ &\times K_{21} \left(\begin{smallmatrix} y_1 \dots y_{i_1-1} s_1 y_{i_1+1} \dots y_{i_q-1} s_q y_{i_q+1} \dots y_m \\ x_1 \dots x_{i_1} \dots x_{i_q} \dots x_m \end{smallmatrix} \right) ds_1 \dots ds_q + \dots \end{aligned}$$

$$+\frac{1}{m!}\int\cdots\int F_m(y_1\cdots y_m; k_{22}) K_{21}(s_1\cdots s_m) ds_1\cdots ds_m.$$

If now we substitute these values for

$F_m(x_1\cdots x_m; k_{12})$ and $F_m(y_1\cdots y_m; k_{21})$ in (4) we get another

formula for $F_o\left[\begin{smallmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{smallmatrix}\right]$, viz:

$$\begin{aligned} (5) \quad F_o\left[\begin{smallmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{smallmatrix}\right] &= F_o(k_{11}) F_o(k_{22}) - \iint \left[F_o(k_{11}) K_{12}(x_1 y_1) \right. \\ &+ \int F_1(x_1; k_{11}) K_{12}(s_1) ds_1 \left. \right] \left[F_o(k_{22}) K_{21}(y_1) \right. \\ &+ \int F_1(y_1; k_{22}) K_{21}(s_1) ds_1 \left. \right] dx_1 dy \\ &+ \frac{1}{(2!)^2} \iiint \left[F_o(k_{11}) K_{12}(x_1 x_2) \right. \\ &+ \int \sum_{i_1=1}^2 F_1(x_{i_1}; k_{11}) K_{12}(x_1 s_1) ds_1 \\ &+ \frac{1}{2!} \iint F_2(x_1 x_2; k_{11}) K_{12}(s_1 s_2) ds_1 ds_2 \left. \right] \left[F_o(k_{22}) K_{21}(y_1 y_2) \right. \\ &+ \int \sum_{i=1}^2 F_1(y_i; k_{22}) K_{21}(y_1 s_1) ds_1 \\ &+ \frac{1}{2!} \iint F_2(y_1 y_2; k_{22}) K_{21}(s_1 s_2) ds_1 ds_2 \left. \right] dx_1 dx_2 dy_1 dy_2 + \cdots \\ &+ \frac{(-1)^m}{(m!)^2} \int \cdots \int \left[F_o(k_{11}) K_{12}(x_1 \cdots x_m) \right. \\ &+ \int \sum_{i=1}^m F_1(x_i; k_{11}) K_{12}(x_1 \cdots x_{i-1} s_1 x_{i+1} \cdots x_m) ds_1 + \cdots \\ &+ \frac{1}{(q!)^2} \int \cdots \int \sum_{i_1=1}^m \cdots \sum_{i_q=1}^m F_q(x_{i_1} \cdots x_{i_q}; k_{11}) \\ &\times K_{12}(x_1 \cdots x_{i_1-1} s_1 x_{i_1+1} \cdots x_{i_q-1} s_q x_{i_q+1} \cdots x_m) ds_1 \cdots ds_q + \cdots \\ &+ \frac{1}{m!} \int \cdots \int F_m(x_1 \cdots x_m; k_{11}) K_{12}(s_1 \cdots s_m) ds_1 \cdots ds_m \left. \right] \end{aligned}$$

$$\begin{aligned}
 & \times \left[F_{\circ}(k_{22}) K_{21} \begin{pmatrix} y_1 \dots y_m \\ x_1 \dots x_m \end{pmatrix} \right. \\
 & \quad + \int_{i=1}^m F_1 \begin{pmatrix} y_i \\ s_1 \end{pmatrix} K_{21} \begin{pmatrix} y_1 \dots y_{i-1} s_1 y_{i+1} \dots y_m \\ x_1 \dots x_i \dots x_m \end{pmatrix} ds_1 + \dots \\
 & \quad + \frac{1}{(q!)^2} \int \dots \int_{i_1=1}^m \dots \int_{i_q=1}^m F_q \begin{pmatrix} y_{i_1} \dots y_{i_q} \\ s_1 \dots s_q \end{pmatrix} \\
 & \quad \times K_{21} \begin{pmatrix} y_1 \dots y_{i_1-1} s_1 y_{i_1+1} \dots y_{i_q-1} s_q y_{i_q+1} \dots y_m \\ x_1 \dots x_{i_1} \dots x_{i_q} \dots x_m \end{pmatrix} ds_1 \dots ds_q + \dots \\
 & \quad + \frac{1}{m!} \int \dots \int F_m \begin{pmatrix} y_1 \dots y_m \\ s_1 \dots s_m \end{pmatrix} \\
 & \quad \times K_{21} \begin{pmatrix} s_1 \dots s_m \\ x_1 \dots x_m \end{pmatrix} ds_1 \dots ds_m \left. \right] dx_1 \dots dx_m dy_1 \dots dy_m + \dots.
 \end{aligned}$$

By expanding (3) according to the columns containing $k_{11}(x, y)$ and $k_{21}(x, y)$ we obtain in a similar manner the formula

$$\begin{aligned}
 (4') \quad F_{\circ} \begin{bmatrix} k_{11} k_{12} \\ k_{21} k_{22} \end{bmatrix} &= F_{\circ}(k_{11}) F_{\circ}(k_{22}) + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \\
 & \times \int \dots \int \bar{F}_m \begin{pmatrix} y_1 \dots y_m \\ x_1 \dots x_m, k_{21} \end{pmatrix} \bar{F}_m \begin{pmatrix} x_1 \dots x_m \\ y_1 \dots y_m, k_{22} \end{pmatrix} dx_1 \dots dx_m dy_1 \dots dy_m,
 \end{aligned}$$

where

$$\begin{aligned}
 \bar{F}_m \begin{pmatrix} x_1 \dots x_m \\ y_1 \dots y_m, k_{22} \end{pmatrix} &= K_{12} \begin{pmatrix} x_1 \dots x_m \\ y_1 \dots y_m \end{pmatrix} - \sum_{n=1}^{\infty} \int \dots \int \frac{(-1)^n}{n!} \\
 & \times \begin{vmatrix} k_{12}(x_1 y_1) \dots k_{12}(x_1 y_m) & k_{12}(x_1 s_1) \dots k_{12}(x_1 s_n) \\ \dots & \dots \\ k_{12}(x_m y_1) \dots k_{12}(x_m y_m) & k_{12}(x_m s_1) \dots k_{12}(x_m s_n) \\ k_{22}(s_1 y_1) \dots k_{22}(s_1 y_m) & k_{22}(s_1 s_1) \dots k_{22}(s_1 s_n) \\ \dots & \dots \\ k_{22}(s_n y_1) \dots k_{22}(s_n y_m) & k_{22}(s_n s_1) \dots k_{22}(s_n s_n) \end{vmatrix} ds_1 \dots ds_n
 \end{aligned}$$

and $\bar{F}_m \begin{pmatrix} y_1 \dots y_m \\ x_1 \dots x_m, k_{11} \end{pmatrix}$ is similarly expressed.

If in (4') we place $k_{21}=k_{11}$ and $k_{22}=k_{12}$ we obtain an interesting formula for the Fredholm determinant of the sum of two kernels⁽¹⁾, viz:

$$(6) \quad F_0(k_{11}+k_{12})=F_0(k_{11}) F_0(k_{12})+\sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2} \\ \times \int \dots \int F_m\left(\begin{smallmatrix} x_1 \dots x_m \\ y_1 \dots y_m \end{smallmatrix}; k_{11}\right) F_m\left(\begin{smallmatrix} y_1 \dots y_m \\ x_1 \dots x_m \end{smallmatrix}; k_{12}\right) dx_1 \dots dx_m dy_1 \dots dy_m.$$

This last formula has here been deduced from an expression for $F_0\left[\begin{smallmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{smallmatrix}\right]$. It can in turn be used to obtain another formula for $F_0\left[\begin{smallmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{smallmatrix}\right]$. To do this let us replace $k_{12}(y, x)$ by $\lambda \int k_{12}(y, s) k_a(s, x) ds$ in (6). We then have

$$F_0(k_{11}+\lambda \int k_{12} k_a)=F_0(k_{11}) \cdot F_0(\lambda \int k_{12} k_a)+\sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2} \\ \times \int \dots \int F_m\left(\begin{smallmatrix} x_1 \dots x_m \\ y_1 \dots y_m \end{smallmatrix}; k_{11}\right) \cdot F_m\left(\begin{smallmatrix} y_1 \dots y_m \\ x_1 \dots x_m \end{smallmatrix}; \lambda \int k_{12} k_a\right) dx_1 \dots dx_m dy_1 \dots dy_m.$$

By expanding this expression still further, making frequent use of the reduction formula⁽²⁾

$$\left| \int K'(x_i s_i) K''(s_i y_j) ds_i \right| \\ = \frac{1}{n!} \int \dots \int K'\left(\begin{smallmatrix} x_1 \dots x_n \\ s_1 \dots s_n \end{smallmatrix}\right) K''\left(\begin{smallmatrix} s_1 \dots s_n \\ y_1 \dots y_n \end{smallmatrix}\right) ds_1 \dots ds_n, \quad \begin{matrix} i=1, 2, 3, \dots, n, \\ j=1, 2, 3, \dots, n, \end{matrix}$$

and collecting terms according to powers of λ we obtain a formula for $F_0(k_{11}+\lambda \int k_{12} k_a)$, viz:

$$(7) \quad F_0(k_{11}+\lambda \int k_{12} k_a)=F_0(k_{11}) \\ -\lambda \iint \left[F_0(k_{11}) k_{12}(x_1 s_1) + \int F_1\left(\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}; k_{11}\right) K_{12}\left(\begin{smallmatrix} y_1 \\ s_1 \end{smallmatrix}\right) dy_1 \right]^k K_a(s_1 x_1) dx_1 ds_1 \\ + \frac{\lambda^2}{(2!)^2} \iiint \left[F_0(k_{11}) K_{12}\left(\begin{smallmatrix} x_1 x_2 \\ s_1 s_2 \end{smallmatrix}\right) + \int \sum_{i=1}^2 F_1\left(\begin{smallmatrix} x_i \\ y_i \end{smallmatrix}; k_{11}\right) K_{12}\left(\begin{smallmatrix} y_i x_i \\ s_1 s_2 \end{smallmatrix}\right) dy_i \right. \\ \left. + \frac{1}{2!} \iint F_2\left(\begin{smallmatrix} x_1 y_1 \\ x_2 y_2 \end{smallmatrix}; k_{11}\right) K_{12}\left(\begin{smallmatrix} y_1 y_2 \\ s_1 s_2 \end{smallmatrix}\right) dy_1 dy_2 \right] K_a\left(\begin{smallmatrix} s_1 s_2 \\ x_1 x_2 \end{smallmatrix}\right) dx_1 dx_2 ds_1 ds_2$$

(1) Goursat, Ann. de la Faculté des Sciences de Toulouse, Series 2, Vol. 10 (1908), p. 27, has obtained a special case of this formula viz: where $k_1(x, y)$ and $k_2(x, y)$ are orthogonal kernels, in which case it reduces to $F_0(k_1+k_2)=F_0(k_1) \cdot F_0(k_2)$.

(2) Cf. Schur, Mathematische Annalen, vol. 67 (1909) p. 319; also Landsberg, ibid, vol. 69 (1910) p. 231.

$$\begin{aligned}
 & + \cdots + \frac{(-\lambda)^m}{(m!)^2} \int \cdots \int \left[F_0(k_{11}) K_{12} \left(\begin{smallmatrix} x_1 \cdots x_m \\ s_1 \cdots s_m \end{smallmatrix} \right) \right. \\
 & + \int \sum_{i=1}^m F_1 \left(\begin{smallmatrix} x_i \\ y_1 \end{smallmatrix} ; k_{11} \right) K_{12} \left(\begin{smallmatrix} x_1 \cdots x_{i-1} y_1 x_{i+1} \cdots x_m \\ s_1 \cdots s_i \cdots s_m \end{smallmatrix} \right) dy_1 + \cdots \\
 & + \frac{1}{(q!)^2} \int \cdots \int \sum_{i_1=1}^m \cdots \sum_{i_q=1}^m F_q \left(\begin{smallmatrix} x_{i_1} \cdots x_{i_q} \\ y_1 \cdots y_q \end{smallmatrix} ; k_{11} \right) \\
 & \times K_{12} \left(\begin{smallmatrix} x_1 \cdots x_{i_1-1} y_1 x_{i_1} \cdots x_{i_q-1} y_q x_{i_q+1} \cdots x_m \\ s_1 \cdots s_{i_1} \cdots s_{i_q} \cdots s_m \end{smallmatrix} \right) dy_1 \cdots dy_q + \cdots \\
 & + \frac{1}{m!} \int \cdots \int F_m \left(\begin{smallmatrix} x_1 \cdots x_m \\ y_1 \cdots y_m \end{smallmatrix} ; k_{11} \right) K_{12} \left(\begin{smallmatrix} y_1 \cdots y_m \\ s_1 \cdots s_m \end{smallmatrix} \right) dy_1 \cdots dy_m \Big] \\
 & \times K_a \left(\begin{smallmatrix} s_1 \cdots s_m \\ x_1 \cdots x_m \end{smallmatrix} \right) dx_1 \cdots dx_m ds_1 \cdots ds_m + \cdots
 \end{aligned}$$

Suppose $F_0(k_{22}) \neq 0$ and place

$$k_a(y_1 x_1) = k_{21}(y_1 x_1) + \int K_{22}(y_1 s_1) k_{21}(s_1 x_1) ds_1,$$

where $K_{22}(y_1 s_1)$ is the reciprocal of $k_{22}(y_1 s_1)$, i. e.

$$K_{22}(y_1 s_1) = \frac{F_1 \left(\begin{smallmatrix} y_1 \\ s_1 \end{smallmatrix} ; k_{22} \right)}{F_0(k_{22})}.$$

If we make use of a second reduction formula⁽¹⁾

$$\frac{F_q \left(\begin{smallmatrix} y_1 \cdots y_q \\ s_1 \cdots s_q \end{smallmatrix} ; k_{22} \right)}{F_0(k_{22})} = \left| \begin{array}{cc} \frac{F_1 \left(\begin{smallmatrix} y_1 \\ s_1 \end{smallmatrix} ; k_{22} \right)}{F_0(k_{22})} & \frac{F_1 \left(\begin{smallmatrix} y_1 \\ s_q \end{smallmatrix} ; k_{22} \right)}{F_0(k_{22})} \\ \cdots & \cdots \\ \frac{F_1 \left(\begin{smallmatrix} y_q \\ s_1 \end{smallmatrix} ; k_{22} \right)}{F_0(k_{22})} & \frac{F_1 \left(\begin{smallmatrix} y_q \\ s_q \end{smallmatrix} ; k_{22} \right)}{F_0(k_{22})} \end{array} \right|$$

and also the formula for $|\int K' K''|$ given above, $K_a \left(\begin{smallmatrix} y_1 \cdots y_q \\ s_1 \cdots s_q \end{smallmatrix} \right)$ can be reduced to

(1) Cf. Plemelj, Monatshefte für Mathematik und Physik, Vol. 15 (1904) p. 101. See also Platrier, loc. cit. p. 249.

$$\begin{aligned}
& K_{21} \left(\begin{matrix} y_1 \cdots y_m \\ x_1 \cdots x_m \end{matrix} \right) + \int \sum_{i=1}^m \frac{F_1 \left(\begin{matrix} y_i \\ s_1 \end{matrix}; k_{22} \right)}{F_0(k_{22})} K_{21} \left(\begin{matrix} y_1 \cdots y_{i-1} s_i y_{i+1} \cdots y_m \\ x_1 \cdots x_i \cdots x_m \end{matrix} \right) ds_i + \cdots \\
& + \frac{1}{(q!)^2} \int \cdots \int \sum_{i_1=1}^m \cdots \sum_{i_q=1}^m \frac{F_q \left(\begin{matrix} y_{i_1} \cdots y_{i_q} \\ s_1 \cdots s_q \end{matrix}; k_{22} \right)}{F_0(k_{22})} \\
& \times K_{21} \left(\begin{matrix} y_1 \cdots y_{i_1-1} s_{i_1} y_{i_1+1} \cdots y_{i_q-1} s_{i_q} y_{i_q+1} \cdots y_m \\ x_1 \cdots x_{i_1} \cdots x_{i_q} \cdots x_m \end{matrix} \right) ds_1 \cdots ds_q + \cdots \\
& + \frac{1}{m!} \int \cdots \int \frac{F_m \left(\begin{matrix} y_1 \cdots y_m \\ s_1 \cdots s_m \end{matrix}; k_{22} \right)}{F_0(k_{22})} K_{21} \left(\begin{matrix} s_1 \cdots s_m \\ x_1 \cdots x_m \end{matrix} \right) ds_1 \cdots ds_m.
\end{aligned}$$

Substituting this expression for $K_{21} \left(\begin{matrix} y_1 \cdots y_q \\ s_1 \cdots s_q \end{matrix} \right)$ in (7) and comparing the result with formula (5) we see that

$$F_0(k_{11} + \int k_{12}(k_{21} + \int K_{22} k_{21})) = \frac{F_0 \left[\begin{matrix} k_{11} k_{12} \\ k_{21} k_{22} \end{matrix} \right]}{F_0(k_{22})}.$$

Hence we have the formula

$$(8) \quad F_0 \left[\begin{matrix} k_{11} k_{12} \\ k_{21} k_{22} \end{matrix} \right] = F_0(k_{11} + \int k_{12}(k_{21} + \int K_{22} k_{21})) F_0(k_{22}),$$

where $K_{22}(x, y)$ is the reciprocal of $k_{22}(x, y)$.

Again, if we apply the first reduction formula above to

$$F_0(\int k' k'') = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \int \cdots \int |\int k' k''| dx_1 \cdots dx_n$$

we obtain

$$\begin{aligned}
(9) \quad F_0(\int k' k'') &= 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2} \\
&\times \int \cdots \int K' \left(\begin{matrix} x_1 \cdots x_m \\ y_1 \cdots y_m \end{matrix} \right) K'' \left(\begin{matrix} y_1 \cdots y_m \\ x_1 \cdots x_m \end{matrix} \right) dx_1 \cdots dx_m dy_1 \cdots dy_m.
\end{aligned}$$

Suppose $F_0(k_{11}) \neq 0$ and $F_0(k_{22}) \neq 0$. Then from (9) it follows that for

$$K'(x, y) = k_{12}(x, y) + \int K_{11}(x, s) k_{12}(s, y) ds$$

and

$$K''(x, y) = k_{21}(x, y) + \int K_{22}(x, s) k_{21}(s, y) ds,$$

where $K_{11}(x, y)$ and $K_{22}(x, y)$ are the reciprocals of $k_{11}(x, y)$ and $k_{22}(x, y)$ respectively, i. e. •

$$K_{11}(x, y) = \frac{F_1\left(\begin{smallmatrix} x \\ y \end{smallmatrix}; k_{11}\right)}{F_0(k_{11})}$$

and

$$K_{22}(x, y) = \frac{F_1\left(\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}; k_{22}\right)}{F_0(k_{22})},$$

we have after applying the second reduction formula above and comparing the result with (5)

$$F_0(\int (k_{12} + \int K_{11} k_{12})(k_{21} + \int K_{22} k_{21})) = \frac{F_0\left[\begin{smallmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{smallmatrix}\right]}{F_0(k_{11}) F_0(k_{22})};$$

i. e. we have still another formula for $F_0\left[\begin{smallmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{smallmatrix}\right]$, viz :

$$(10) \quad F_0\left[\begin{smallmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{smallmatrix}\right] = F_0(\int (k_{12} + \int K_{11} k_{12})(k_{21} + \int K_{22} k_{21})) \cdot F_0(k_{11}) \cdot F_0(k_{22}).$$

We have then the following formulæ for the development of the Fredholm determinants of the various combinations of kernels which we have considered.

$$\begin{aligned} \text{I. } F_0(k_1 + k_2) &= F_0(k_1) F_0(k_2) + \sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2} \\ &\times \int \dots \int F_m\left(\begin{smallmatrix} x_1 \dots x_m \\ y_1 \dots y_m \end{smallmatrix}; k_1\right) F_m\left(\begin{smallmatrix} y_1 \dots y_m \\ x_1 \dots x_m \end{smallmatrix}; k_2\right) dx_1 \dots dx_m dy_1 \dots dy_m. \end{aligned}$$

$$\begin{aligned} \text{II. } F_0(k_1 + \lambda \int k_2 k_{11}) &= F_0(k_1) - \lambda \iint \left[F_0(k_1) k_2(x_1 s_1) \right. \\ &+ \left. \int F_1\left(\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}; k_1\right) K_2\left(\begin{smallmatrix} y_1 \\ s_1 \end{smallmatrix}\right) dy_1 \right] k_2(s_1 x_1) dx_1 ds_1 \\ &+ \frac{\lambda^2}{(2!)^2} \iiint F_2(k_1) K_2\left(\begin{smallmatrix} x_1 & x_2 \\ s_1 & s_2 \end{smallmatrix}\right) + \int \sum_{i=1}^2 F_1\left(\begin{smallmatrix} x_i \\ y_1 \end{smallmatrix}; k_1\right) K_2\left(\begin{smallmatrix} y_1 & x_i \\ s_1 & s_1 \end{smallmatrix}\right) dy_1 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2!} \int \int F_2 \left(\begin{smallmatrix} x_1 x_2 \\ y_1 y_2 \end{smallmatrix}; k_1 \right) K_2 \left(\begin{smallmatrix} y_1 y_2 \\ s_1 s_2 \end{smallmatrix} \right) dy_1 dy_2 \left] K_a \left(\begin{smallmatrix} s_1 s_2 \\ x_1 x_2 \end{smallmatrix} \right) dx_1 dx_2 ds_1 ds_2 \right. \\
& + \dots + \frac{(-\lambda)^m}{(m!)} \int \dots \int \left[F_0(k_1) K_2 \left(\begin{smallmatrix} x_1 \dots x_m \\ s_1 \dots s_m \end{smallmatrix} \right) \right. \\
& + \sum_{i=1}^m F_1 \left(\begin{smallmatrix} x_i \\ y_1 \end{smallmatrix}; k_1 \right) K_2 \left(\begin{smallmatrix} x_1 \dots x_{i-1} y_1 x_{i+1} \dots x_m \\ s_1 \dots s_{i-1} s_i \dots s_m \end{smallmatrix} \right) dy_1 + \dots \\
& + \frac{1}{(q!)^2} \int \dots \int \sum_{i_1=1}^m \dots \sum_{i_q=1}^m F_q \left(\begin{smallmatrix} x_{i_1} \dots x_{i_q} \\ y_1 \dots y_q \end{smallmatrix}; k_1 \right) \\
& \times K_2 \left(\begin{smallmatrix} x_1 \dots x_{i_1-1} y_1 x_{i_1+1} \dots x_{i_q-1} y_q x_{i_q+1} \dots x_m \\ s_1 \dots s_{i_1-1} s_{i_1} \dots s_{i_q-1} s_{i_q} \dots s_m \end{smallmatrix} \right) dy_1 \dots dy_q + \dots \\
& + \frac{1}{m!} \int \dots \int F_m \left(\begin{smallmatrix} x_1 \dots x_m \\ y_1 \dots y_m \end{smallmatrix}; k_1 \right) K_2 \left(\begin{smallmatrix} y_1 \dots y_m \\ s_1 \dots s_m \end{smallmatrix} \right) dy_1 \dots dy_m \left] \right. \\
& \times K_a \left(\begin{smallmatrix} s_1 \dots s_m \\ x_1 \dots x_m \end{smallmatrix} \right) dx_1 \dots dx_m ds_1 \dots ds_m.
\end{aligned}$$

$$\begin{aligned}
\text{III. } F_0 \left[\begin{smallmatrix} k_{11} k_{12} \\ k_{21} k_{22} \end{smallmatrix} \right] &= F_0(k_{11}) F_0(k_{22}) + \sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2} \\
&\times \int \dots \int F_m \left(\begin{smallmatrix} x_1 \dots x_m \\ y_1 \dots y_m \end{smallmatrix}; \begin{smallmatrix} k_{12} \\ k_{11} \end{smallmatrix} \right) F_m \left(\begin{smallmatrix} y_1 \dots y_m \\ x_1 \dots x_m \end{smallmatrix}; \begin{smallmatrix} k_{21} \\ k_{22} \end{smallmatrix} \right) dx_1 \dots dx_m dy_1 \dots dy_m
\end{aligned}$$

and similar one,

$$\begin{aligned}
\text{III'. } F_0 \left[\begin{smallmatrix} k_{11} k_{11} \\ k_{21} k_{22} \end{smallmatrix} \right] &= F_0(k_{11}) F_0(k_{22}) + \sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2} \\
&\times \int \dots \int \bar{F}_m \left(\begin{smallmatrix} y_1 \dots y_m \\ x_1 \dots x_m \end{smallmatrix}; \begin{smallmatrix} k_{21} \\ k_{11} \end{smallmatrix} \right) \bar{F}_m \left(\begin{smallmatrix} x_1 \dots x_m \\ y_1 \dots y_m \end{smallmatrix}; \begin{smallmatrix} k_{12} \\ k_{22} \end{smallmatrix} \right) dx_1 \dots dx_m dy_1 \dots dy_m.
\end{aligned}$$

$$\begin{aligned}
\text{IV. } F_0 \left[\begin{smallmatrix} k_{11} k_{12} \\ k_{21} k_{22} \end{smallmatrix} \right] &= F_0(k_{11}) F_0(k_{22}) + \sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2} \int \dots \int \left[F_0(k_{11}) K_{12} \left(\begin{smallmatrix} x_1 \dots x_m \\ y_1 \dots y_m \end{smallmatrix} \right) \right. \\
& + \int \sum_{i=1}^m F_1 \left(\begin{smallmatrix} x_i \\ s_1 \end{smallmatrix}; k_{11} \right) K_{12} \left(\begin{smallmatrix} x_1 \dots x_{i-1} s_1 x_{i+1} \dots x_m \\ y_1 \dots y_{i-1} y_i \dots y_m \end{smallmatrix} \right) ds_1 + \dots \\
& + \frac{1}{(q!)^2} \int \dots \int \sum_{i_1=1}^m \dots \sum_{i_q=1}^m F_q \left(\begin{smallmatrix} x_{i_1} \dots x_{i_q} \\ s_1 \dots s_q \end{smallmatrix}; k_{11} \right) \\
& \times K_{12} \left(\begin{smallmatrix} x_1 \dots x_{i_1-1} s_1 x_{i_1+1} \dots x_{i_q-1} s_q x_{i_q+1} \dots x_m \\ y_1 \dots y_{i_1-1} y_{i_1} \dots y_{i_q-1} y_{i_q} \dots y_m \end{smallmatrix} \right) ds_1 \dots ds_q \\
& + \dots + \frac{1}{m!} \int \dots \int F_m \left(\begin{smallmatrix} x_1 \dots x_m \\ s_1 \dots s_m \end{smallmatrix}; k_{11} \right) K_{12} \left(\begin{smallmatrix} s_1 \dots s_m \\ y_1 \dots y_m \end{smallmatrix} \right) ds_1 \dots ds_m \left] \right.
\end{aligned}$$

$$\begin{aligned}
 & \times \left[F_0'(k_{22}) K_{21} \left(y_1 \cdots y_n; x_1 \cdots x_n \right) + \int \sum_{i=1}^m F_i' \left(y_i; k_{22} \right) K_{21} \left(y_1 \cdots y_{i-1} s_1 y_{i+1} \cdots y_m \right) ds_1 \right. \\
 & + \cdots + \frac{1}{(q!)^2} \int \cdots \int \sum_{i_1=1}^m \cdots \sum_{i_q=1}^m F_q \left(y_{i_1} \cdots y_{i_q}; k_{22} \right) \\
 & \times K_{21} \left(y_1 \cdots y_{i_1-1} s_1 y_{i_1+1} \cdots y_{i_q-1} s_q y_{i_q+1} \cdots y_m \right) ds_1 \cdots ds_q + \cdots \\
 & \left. + \frac{1}{m!} \int \cdots \int F_m \left(y_1 \cdots y_m; k_{22} \right) K_{21} \left(s_1 \cdots s_m \right) ds_1 \cdots ds_m \right] dx_1 \cdots \\
 & dx_m dy_1 \cdots dy_m.
 \end{aligned}$$

$$\text{V. } F_0 \left[\begin{matrix} k_{11} k_{12} \\ k_{21} k_{22} \end{matrix} \right] = F_0(k_{11} + \int k_{12}(k_{21} + \int K_{22} k_{21})) F_0(k_{22}).$$

$$\text{VI. } F_0 \left[\begin{matrix} k_{11} k_{12} \\ k_{21} k_{22} \end{matrix} \right] = F_0 \left(\int (k_{12} + \int K_{11} k_{12})(k_{21} + \int K_{22} k_{21}) \right) F_0(k_{11}) F_0(k_{22}).$$

§ 2. Consider the linear integral expression

$$(1) \quad \varphi_1(x) - \int k_{11}(x, y) \varphi_1(y) dy - \int k_{12}(x, y) \varphi_2(y) dy = f_1(x).$$

We have defined linear dependence as follows: The linear integral expression (1) is said to be linearly dependent if there exists a set of functions $\varphi_1(x)$, $\varphi_2(x)$, not both zero, which satisfies the system of equations:

$$\begin{aligned}
 (2) \quad & \varphi_1(x) - \int k_{11}(x, y) \varphi_1(y) dy - \int k_{12}(x, y) \varphi_2(y) dy = 0 \\
 & \varphi_2(x) - \int k_{21}(x, y) \varphi_1(y) dy - \int k_{22}(x, y) \varphi_2(y) dy = 0
 \end{aligned}$$

for every $k_{21}(x, y)$ and $k_{22}(x, y)$, i.e. if the Fredholm determinant of the system of equations (2) is zero for all $k_{21}(x, y)$ and $k_{22}(x, y)$.

We then have

Theorem I. A necessary and sufficient condition that (1) be linearly dependent is that $k_{11}(x, y)$ and $k_{12}(x, y)$ be such that

$$F_0(k_{11}) = 0$$

and

$$\int \cdots \int F_m \left(x_1 x_2 \cdots x_m; k_{11} \right) K_{12} \left(s_1 s_2 \cdots s_m \right) ds_1 ds_2 \cdots ds_m = 0$$

for $m=1, 2, 3, \dots$.

For, from formula IV of the preceding section it follows at once that if these conditions are fulfilled, then we must have

$$F_0 \begin{bmatrix} k_{11} k_{12} \\ k_{21} k_{22} \end{bmatrix} = 0$$

for every $k_{21}(x, y)$ and $k_{22}(x, y)$, i.e. our condition is sufficient.

To show that the condition is also necessary let us set

$$k_{21}(x, y) = \lambda \bar{k}_{21}(x, y).$$

Then Formula IV gives us $F_0 \begin{bmatrix} k_{11} k_{12} \\ \lambda \bar{k}_{21} k_{22} \end{bmatrix}$ as a power series in λ which by hypothesis would be zero for every value of λ . Hence the coefficient of each power of λ will be equal to zero, i.e.

$$F_0(k_{11}) F_0(k_{22}) = 0,$$

$$\begin{aligned} & \iint \left[F_0(k_{11}) k_{12}(x_1 y_1) + \int F_1 \left(\begin{smallmatrix} x_1 \\ s_1 \end{smallmatrix}; k_{11} \right) K_{12} \left(\begin{smallmatrix} s_1 \\ y_1 \end{smallmatrix} \right) ds_1 \right] \\ & \times \left[F_0(k_{22}) \bar{K}_{21} \left(\begin{smallmatrix} y_1 \\ x_1 \end{smallmatrix} \right) + \int F_1 \left(\begin{smallmatrix} y_1 \\ s_1 \end{smallmatrix}; k_{22} \right) \bar{K}_{21} \left(\begin{smallmatrix} s_1 \\ x_1 \end{smallmatrix} \right) ds_1 \right] dx_1 dy_1 = 0, \end{aligned}$$

.....
.....

$$\begin{aligned} & \int \dots \int \left[F_0(k_{11}) K_{12} \left(\begin{smallmatrix} x_1 \dots x_m \\ y_1 \dots y_m \end{smallmatrix} \right) + \dots \right. \\ & + \frac{1}{(q!)^2} \int \dots \int \sum_{i_1=1}^m \dots \sum_{i_q=1}^m F_q \left(\begin{smallmatrix} x_{i_1} \dots x_{i_q} \\ s_1 \dots s_q \end{smallmatrix}; k_{11} \right) \\ & \times K_{12} \left(\begin{smallmatrix} x_1 \dots x_{i_1-1} s_1 x_{i_1+1} \dots x_{i_q-1} s_q x_{i_q+1} \dots x_m \\ y_1 \dots y_{i_1} \dots y_{i_q} \dots y_m \end{smallmatrix} \right) ds_1 \dots ds_q + \dots \\ & + \frac{1}{m!} \int \dots \int F_m \left(\begin{smallmatrix} x_1 \dots x_m \\ s_1 \dots s_m \end{smallmatrix}; k_{11} \right) K_{12} \left(\begin{smallmatrix} s_1 \dots s_m \\ y_1 \dots y_m \end{smallmatrix} \right) ds_1 \dots ds_m \Big] \\ & \times \left[F_0(k_{22}) \bar{K}_{21} \left(\begin{smallmatrix} y_1 \dots y_m \\ x_1 \dots x_m \end{smallmatrix} \right) + \dots + \frac{1}{(q!)^2} \int \dots \int \sum_{i_1=1}^m \dots \sum_{i_q=1}^m F_q \left(\begin{smallmatrix} y_{i_1} \dots y_{i_q} \\ s_1 \dots s_q \end{smallmatrix}; k_{22} \right) \right. \\ & \times \bar{K}_{21} \left(\begin{smallmatrix} y_1 \dots y_{i_1-1} s_1 y_{i_1+1} \dots y_{i_q-1} s_q y_{i_q+1} \dots y_m \\ x_1 \dots x_{i_1} \dots x_{i_q} \dots x_m \end{smallmatrix} \right) ds_1 \dots ds_q + \dots \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{m!} \int \dots \int F_m \left(\begin{matrix} y_1 \dots y_m \\ s_1 \dots y_m \end{matrix}; k_{22} \right) \bar{K}_{21} \left(\begin{matrix} s_1 \dots s_m \\ x_1 \dots x_m \end{matrix} \right) ds_1 \dots ds_m \Big] \\
 & \quad \times dx_1 \dots dx_m dy_1 \dots dy_m = 0, \\
 & \dots\dots\dots \\
 & \dots\dots\dots
 \end{aligned}$$

If in these equations we set $k_{21}(y, x) = k_{12}(x, y)$ and $k_{22}(y, x) = k_{11}(x, y)$, then since from $\int [\varphi(x)]^2 dx = 0$ it follows that $\varphi(x) = 0$ we get

$$\begin{aligned}
 & F_0(k_{11}) = 0, \\
 & \int F_1 \left(\begin{matrix} x_1 \\ s_1 \end{matrix}; k_{11} \right) K_{12} \left(\begin{matrix} s_1 \\ y_1 \end{matrix} \right) ds_1 = 0, \\
 & \dots\dots\dots \\
 & \dots\dots\dots \\
 & \int \dots \int F_m \left(\begin{matrix} x_1 \dots x_m \\ s_1 \dots s_m \end{matrix}; k_{11} \right) K_{12} \left(\begin{matrix} s_1 \dots s_m \\ y_1 \dots y_m \end{matrix} \right) ds_1 \dots ds_m = 0, \\
 & \dots\dots\dots \\
 & \dots\dots\dots
 \end{aligned}$$

which are the conditions of our theorem. From Formula V of the preceding section we obtain a second condition for linear dependence, viz:

Theorem II. A necessary and sufficient condition that (1) be linearly dependent is that $k_{11}(x, y)$ and $k_{12}(x, y)$ be such that

$$F_0(k_{11} + \int k_{12} k_a) = 0$$

for every $k_a(x, y)$.

For we must have

$$F_0 \left[\begin{matrix} k_{11} k_{12} \\ k_{21} k_{22} \end{matrix} \right] = F_0 \left(k_{11} + \int k_{12} (k_{21} + \int K_{22} k_{21}) \right) F_0(k_{22}) = 0$$

for every $k_{21}(x, y)$ and $k_{22}(x, y)$.

Choose $k_{22}(x, y)$ such that $F_0(k_{22}) \neq 0$. Then we must have

$$F_0(k_{11} + \int k_{12} (k_{21} + \int K_{22} k_{21})) = 0,$$

i.e. if we determine $k_{21}(x, y)$ so that $k_{21} + \int K_{22} k_{21} = k_u$, which is possible

since $F_0(k_{22}) \neq 0$, then we must have $F_0(k_{11} + \int k_{12} k_a) = 0$ for every $k_a(x, y)$. Hence our condition is necessary.

On the other hand, suppose $F_0(k_{11} + \int k_{12} k_a) = 0$ for every $k_a(x, y)$. Then if $F_0(k_{22}) \neq 0$ and we set

$$k_a(x, y) = k_{21}(x, y) + \int K_{22}(x, s) k_{21}(s, y) ds$$

we shall have

$$F_0 \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = 0.$$

Since $F_0 \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$ is continuous when considered as a function of $k_{22}(x, y)$ it follows that it will be zero for every value of $k_{22}(x, y)$. Hence our condition is also sufficient.

From Formula III' of § 1, it is easy to deduce a simple necessary condition for linear dependence in (1) which is not sufficient. We have

Theorem III. A necessary condition that (1) be linearly dependent is that $F_0(k_{11} + \lambda k_{12}) = 0$ for all values of λ .

For we must have

$$\begin{aligned} F_0 \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} &= F_0(k_{11}) F_0(k_{22}) + \sum_{i=1}^m \frac{(-1)^m}{(m!)^2} \\ &\times \int \cdots \int \bar{F}_0(x_1 \cdots x_2; k_{21}) \bar{F}_0(y_1 \cdots y_m; k_{12}) dx_1 \cdots dx_m dy_1 \cdots dy_m = 0 \end{aligned}$$

for every $k_{21}(x, y)$ and $k_{22}(x, y)$. Choose $k_{21}(x, y) = \lambda k_{11}(x, y)$ and $k_{22}(x, y) = \lambda k_{12}(x, y)$. Then we must have

$$\begin{aligned} F_0 \begin{bmatrix} k_{11} & k_{12} \\ \lambda k_{11} & \lambda k_{12} \end{bmatrix} &= F_0(k_{11}) F_0(\lambda k_{12}) + \sum_{i=1}^m \frac{(-\lambda)^m}{(m!)^2} \\ &\times \int \cdots \int F_m(x_1 \cdots x_m; k_{11}) F_m(y_1 \cdots y_m; \lambda k_{12}) dx_1 \cdots dx_m dy_1 \cdots dy_m \\ &= F_0(k_{11} + \lambda k_{12}) = 0 \end{aligned}$$

which shows that our condition is necessary ⁽¹⁾.

⁽¹⁾ This theorem can be easily proved directly. For choose $k_{21}(x, y) = \lambda k_{11}(x, y)$ and $k_{22}(x, y) = \lambda k_{12}(x, y)$. Then the existence of a non-zero solution of the system (2) is equivalent to the existence of a non-zero solution of the single equation

$$\varphi_1(x) - \int (k_{11}(x, y) + \lambda k_{22}(x, y)) \varphi_1(y) dy = 0,$$

i.e. we must have $F_0(k_{11} + \lambda k_{22}) = 0$ for all values of λ .

On the other hand it is possible to obtain a simple condition for linear dependence which is sufficient without being necessary.

Suppose $F_0(k_{11})=0$ and that $F_m\left(\begin{smallmatrix} x_1 \cdots x_m \\ y_1 \cdots y_m \end{smallmatrix}; k_{11}\right)$ is the first non-zero minor of the Fredholm determinant of $k_{11}(x, y)$. Then any solution $\phi_i(s)$ of the integral equation

$$(3) \quad \phi(s) - \int \phi(y) k_{11}(y, s) dy = 0$$

will be of the form⁽¹⁾

$$\phi_i(s) = \sum_{i=1}^m \alpha_i F_m\left(\begin{smallmatrix} x_1^\circ \cdots x_i^\circ \cdots x_m^\circ \\ s_1^\circ \cdots s_{i-1}^\circ s s_{i+1}^\circ \cdots s_m^\circ \end{smallmatrix}; k_{11}\right).$$

Assume now that we have

$$\int F_m\left(\begin{smallmatrix} x_1 \cdots x_m \\ s_1 \cdots s_m \end{smallmatrix}; k_{11}\right) K_{12}\left(\begin{smallmatrix} s_1 \\ y_1 \end{smallmatrix}\right) ds_1 = 0$$

for all values of $x_i (i=1, 2, \dots, m)$; $s_j (j=2, 3, \dots, m)$; and y_1 .

It follows then that

$$\int \cdots \int F_m\left(\begin{smallmatrix} x_1 \cdots x_m \\ s_1 \cdots s_m \end{smallmatrix}; k_{11}\right) K_{12}\left(\begin{smallmatrix} s_1 \cdots s_m \\ y_1 \cdots y_m \end{smallmatrix}\right) ds_1 \cdots ds_m = 0$$

for all values of x_i and $y_i (i=1, 2, \dots, m)$. This is obvious if $K_{12}\left(\begin{smallmatrix} s_1 \cdots s_m \\ y_1 \cdots y_m \end{smallmatrix}\right)$

be expanded according to the minors of the elements of the first column.

Also if we multiply the formula⁽²⁾

$$\begin{aligned} F_{m+1}\left(\begin{smallmatrix} x x_1 \cdots x_m \\ s s_1 \cdots s_m \end{smallmatrix}; k_{11}\right) - \int k_{11}(t, s) F_{m+1}\left(\begin{smallmatrix} x x_1 \cdots x_m \\ t s_1 \cdots s_m \end{smallmatrix}; k_{11}\right) dt \\ \equiv k_{11}(x, s) F_m\left(\begin{smallmatrix} x_1 \cdots x_m \\ s_1 \cdots s_m \end{smallmatrix}; k_{11}\right) - \sum_{i=1}^m k_{11}(x_i, s) F_m\left(\begin{smallmatrix} x_1 \cdots x_{i-1} x_{i+1} \cdots x_m \\ s_1 \cdots s_{i-1} s_{i+1} \cdots s_m \end{smallmatrix}; k_{11}\right) \end{aligned}$$

through by $K_{12}\left(\begin{smallmatrix} s_1 \cdots s_m \\ y_1 \cdots y_m \end{smallmatrix}\right) ds_1 \cdots ds_m$ and integrate it follows that

$$\begin{aligned} \int \cdots \int F_{m+1}\left(\begin{smallmatrix} x x_1 \cdots x_m \\ s s_1 \cdots s_m \end{smallmatrix}; k_{11}\right) K_{12}\left(\begin{smallmatrix} s_1 \cdots s_m \\ y_1 \cdots y_m \end{smallmatrix}\right) ds_1 \cdots ds_m \\ - \int k_{11}(t, s) \left[\int \cdots \int F_{m+1}\left(\begin{smallmatrix} x x_1 \cdots x_m \\ t s_1 \cdots s_m \end{smallmatrix}; k_{11}\right) K_{12}\left(\begin{smallmatrix} s_1 \cdots s_m \\ y_1 \cdots y_m \end{smallmatrix}\right) ds_1 \cdots ds_m \right] dt \equiv 0 \end{aligned}$$

(1) Cf. for instance, Heywood and Fréchet, Équation de Fredholm, p. 71.

(2) Cf. Heywood and Fréchet, loc. cit., p. 70.

i.e.
$$\int \cdots \int F_{m+1} \left(\begin{smallmatrix} x_1^0 & x_1^0 & \cdots & x_m^0 \\ s & s_1 & \cdots & s_m \end{smallmatrix}; k_{11} \right) K_{12} \left(\begin{smallmatrix} s_1 & \cdots & s_m \\ y_1^0 & \cdots & y_m^0 \end{smallmatrix} \right) ds_1 \cdots ds_m$$

is a solution of (3). Hence after placing

$$\begin{aligned} & \int \cdots \int F_{m+1} \left(\begin{smallmatrix} x_1 & x_1 & \cdots & x_m \\ s & s_1 & \cdots & s_m \end{smallmatrix}; k_{11} \right) K_{12} \left(\begin{smallmatrix} s_1 & \cdots & s_m \\ y_1 & \cdots & y_m \end{smallmatrix} \right) ds_1 \cdots ds_m \\ &= \int \cdots \int F_{m+1} \left(\begin{smallmatrix} x_1 & x_2 & \cdots & x_{m+1} \\ t_1 & s_2 & \cdots & s_{m+1} \end{smallmatrix}; k_{11} \right) K_{12} \left(\begin{smallmatrix} s_2 & \cdots & s_{m+1} \\ y_2 & \cdots & y_{m+1} \end{smallmatrix} \right) ds_2 \cdots ds_{m+1} \end{aligned}$$

we have

$$\begin{aligned} & \int \cdots \int F_{m+1} \left(\begin{smallmatrix} x_1 & x_2 & \cdots & x_{m+1} \\ t & s_2 & \cdots & s_{m+1} \end{smallmatrix}; k_{11} \right) K_{12} \left(\begin{smallmatrix} s_2 & \cdots & s_{m+1} \\ y_2 & \cdots & y_{m+1} \end{smallmatrix} \right) ds_2 \cdots ds_{m+1} \\ &= \sum_{i=1}^m \hat{\xi}_i F_m \left(\begin{smallmatrix} x_1 & \cdots & x_i & \cdots & x_m \\ s_1 & \cdots & s_{i-1} & t & s_{i+1} & \cdots & s_m \end{smallmatrix}; k_{11} \right), \end{aligned}$$

where the functions $\hat{\xi}_i$ do not contain the variable t . Multiplying this equality through by $K_{12} \left(\begin{smallmatrix} t \\ y_1 \end{smallmatrix} \right) dt$ and integrating we get

$$\begin{aligned} & \int \cdots \int F_{m+1} \left(\begin{smallmatrix} x_1 & x_2 & \cdots & x_{m+1} \\ t & s_2 & \cdots & s_{m+1} \end{smallmatrix}; k_{11} \right) \\ & \quad \times K_{12} \left(\begin{smallmatrix} s_2 & \cdots & s_{m+1} \\ y_2 & \cdots & y_{m+1} \end{smallmatrix} \right) K_{12} \left(\begin{smallmatrix} t \\ y_1 \end{smallmatrix} \right) ds_2 \cdots ds_{m+1} dt \equiv 0 \end{aligned}$$

in $x_1, x_2, \cdots, x_{m+1}; y_1, y_2, \cdots, y_{m+1}$.

From this it follows that

$$\int \cdots \int F_{m+1} \left(\begin{smallmatrix} x_1 & x_2 & \cdots & x_{m+1} \\ s_1 & s_2 & \cdots & s_{m+1} \end{smallmatrix}; k_{11} \right) K_{12} \left(\begin{smallmatrix} s_1 & s_2 & \cdots & s_{m+1} \\ y_1 & y_2 & \cdots & y_{m+1} \end{smallmatrix} \right) ds_1 \cdots ds_{m+1} \equiv 0$$

in $x_1, x_2, \cdots, x_{m+1}, y_1; y_2, \cdots, y_{m+1}$, which is obvious if $K_{12} \left(\begin{smallmatrix} s_1 & s_2 & \cdots & s_{m+1} \\ y_1 & y_2 & \cdots & y_{m+1} \end{smallmatrix} \right)$ be expanded according to the elements of the first column.

By continued repetition of the above process we can show that

$$\begin{aligned} & \int \cdots \int F_{m+2} \left(\begin{smallmatrix} x_1 & \cdots & x_{m+2} \\ s_1 & \cdots & s_{m+2} \end{smallmatrix}; k_{11} \right) K_{12} \left(\begin{smallmatrix} s_1 & \cdots & s_{m+2} \\ y_1 & \cdots & y_{m+2} \end{smallmatrix} \right) ds_1 \cdots ds_{m+2} \equiv 0 \\ & \cdots \cdots \cdots \\ & \cdots \cdots \cdots \end{aligned}$$

in x and y , i.e. we shall have from Formula IV, § 1, $F_{\circ} \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = 0$

for all values of $k_{21}(x, y)$ and $k_{22}(x, y)$.

We then have proved

Theorem IV. A sufficient condition for linear dependence in (1) is that $k_{11}(x, y)$ and $k_{12}(x, y)$ be such that $F_o(k_{11})=0$ and

$\int F_m \left(\begin{smallmatrix} x_1 & \dots & x_m \\ s_1 & \dots & s_m \end{smallmatrix}; k_{11} \right) K_{12} \left(\begin{smallmatrix} s_1 \\ y_1 \end{smallmatrix} \right) ds_1 = 0$, where $F_m \left(\begin{smallmatrix} x_1 & \dots & x_m \\ s_1 & \dots & s_m \end{smallmatrix}; k_{11} \right)$ is the first minor of the Fredholm determinant of $k_{11}(x, y)$ which is not identically zero.

§ 3. At the end of § 2 we found that if $F_o(k_{11})=0$ and

$F_m \left(\begin{smallmatrix} x_1 & \dots & x_m \\ y_1 & \dots & y_m \end{smallmatrix}; k_{11} \right)$ is the first minor of the Fredholm determinant of $k_{11}(x, y)$ which is not identically zero, then $F_o(k_{11})=0$ and

$\int F_m \left(\begin{smallmatrix} x_1 & \dots & x_m \\ s_1 & \dots & s_m \end{smallmatrix}; k_{11} \right) k_{12}(s_1, y_1) ds_1 = 0$ for every $x_1, \dots, x_m; s_1, \dots, s_m;$

and y_1 , are sufficient conditions for linear dependence in (1). If we recall that for any set of values $x_1^o, \dots, x_m^o; y_1^o, \dots, y_{i-1}^o, y_{i+1}^o, \dots, y_m^o$,

$F_m \left(\begin{smallmatrix} x_1^o & \dots & x_i^o & \dots & x_m^o \\ y_1^o & \dots & y_{i-1}^o & y & y_{i+1}^o & \dots & y_m^o \end{smallmatrix}; k_{11} \right)$ is a solution of the homogeneous equation

$$\phi_1(y) - \int \phi_1(x) k_{11}(x, y) dx = 0,$$

this result suggests the question whether the existence of a solution of the two equations

$$\phi_1(x) - \int \phi_1(y) k_{11}(y, x) dy = 0,$$

(4)

$$\int \phi_1(y) k_{12}(y, x) dy = 0$$

is not a sufficient and even a necessary condition for linear dependence in (1). It turns out that it is both necessary and sufficient, i.e. we have a theorem analogous to Theorem I of the introduction, viz:

Theorem I. A necessary and sufficient condition that (1) be linearly dependent is that there exists a function $\phi_1(x)$ not zero such that the relations (4) are satisfied.

Let us assume that $\phi_1(x) = \phi_1^o(x)$ is a common solution of the equations (4). Then if we choose this $\phi_1^o(x)$ together with $\phi_2(x) = 0$ we shall have for every $k_{21}(x, y)$ and $k_{22}(x, y)$ a non-zero solution of the system

$$(5) \quad \begin{aligned} \phi_1(x) - \int \phi_1(y) k_{11}(y, x) dy - \int \phi_2(y) k_{21}(y, x) dy &= 0, \\ \phi_2(x) - \int \phi_1(y) k_{12}(y, x) dy - \int \phi_2(y) k_{22}(y, x) dy &= 0, \end{aligned}$$

which is the homogeneous adjoint system of (2). But the existence of a non-zero solution of (5) implies the existence of a non-zero solution of (2). The expression (1) is then linearly dependent and we have shown that the condition stated in our theorem is sufficient.

On the other hand, the condition is also necessary, i.e. there exists a solution of the system of equations (4) which is not identically zero.

Let us first show that we must have $F'_0(k_{11})=0$. Since $k_{21}(x, y)$ and $k_{22}(x, y)$ are arbitrary, let us choose $k_{21}(x, y)=0$. Now if the system (2) has a solution other than $\phi_1(x)=\phi_2(x)=0$, it is also true of the system (5). Then the first equation of (5) gives

$$(6) \quad \phi_1(x) - \int \phi_1(y) k_{11}(y, x) dy = 0.$$

There are now two cases to be considered:

$$\begin{aligned} (a) \quad & F'_0(k_{11}) \neq 0, \\ (b) \quad & F'_0(k_{11}) = 0. \end{aligned}$$

Suppose that (a) is true. Then the only possible solution of the equation (6) would be $\phi_1(x)=0$ and the second equation of (5) would give

$$\phi_2(x) - \int \phi_2(y) k_{22}(y, x) dy = 0.$$

Consequently, since by hypothesis there exists a non-zero solution of (5), it would be necessary to have $F'_0(k_{22})=0$, for every $k_{22}(x, y)$, which is impossible, i.e. $k_{11}(x, y)$ must be such that $F'_0(k_{11})=0$ and there exist solutions of the equation (6).

Now let us show that some solution of equation (6) satisfies also the second of the equations (4).

Suppose that the m functions $\phi_1^{(1)}(x)$, $\phi_1^{(2)}(x)$, ..., $\phi_1^{(m)}(x)$ constitute a complete system of linearly independent solutions of the equation (6) and let us substitute there $\phi_1^{(i)}(y)$, ($i=1, 2, \dots, m$) for $\phi_1(y)$ in the second equation of (4) and place

$$\phi_2^{(i)}(x) = \int \phi_1^{(i)}(y) k_{12}(y, x) dy \quad i=1, 2, \dots, m.$$

Then either

(a) the $\phi_2^{(i)}(x)$ are linearly independent,

or

(b) the $\phi_2^{(i)}(x)$ are linearly dependent.

If the $\phi_2^{(i)}(x)$ are linearly independent, there exists an equivalent set of functions $\xi^{(1)}(x), \xi^{(2)}(x), \dots, \xi^{(m)}(x)$, which are normed and orthogonal and such that the $\xi^{(i)}(x)$ are linear combinations of the $\phi_2^{(i)}(x)$ and conversely, the $\phi_2^{(i)}(x)$ are linear combinations of the $\xi^{(i)}(x)$.

If now the m normed and orthogonal functions $\varphi_1^{(1)}(x), \varphi_2^{(2)}(x), \dots, \varphi_m^{(m)}(x)$, constitute a complete system of linear independent solutions of the equation

$$(7) \quad \varphi_1(x) - \int k_{11}(x, y) \varphi_1(y) dy = 0,$$

let us choose ⁽¹⁾

$$k_{21}(x, y) = \sum_{i=1}^m \xi^{(i)}(x) \varphi_1^{(i)}(y)$$

and

$$k_{22}(x, y) = \sum_{i=1}^m \xi^{(i)}(x) \xi^{(i)}(y).$$

Then obviously the $\varphi_2^{(i)}(x)$ will constitute a complete linearly independent set of solutions of the equation

$$(8) \quad \varphi_2(x) - \int k_{22}(x, y) \varphi_2(y) dy = 0.$$

Then, if the system

$$(5') \quad \begin{aligned} \phi_1(x) - \int \phi_1(y) k_{11}(y, x) dy - \int \phi_2(y) \sum_{i=1}^m \xi^{(i)}(y) \varphi_1^{(i)}(x) dy &= 0, \\ \phi_2(x) - \int \phi_1(y) k_{12}(y, x) dy - \int \phi_2(y) \sum_{i=1}^m \xi^{(i)}(y) \xi^{(i)}(x) dy &= 0 \end{aligned}$$

is to have a solution not identically zero, it must be possible to solve the first equation for $\phi_1(x)$ in terms of $\phi_2(x)$ and then substitute this value of $\phi_1(x)$ in the second equation and solve for $\phi_2(x)$.

Since $R_o(k_{11})=0$, it is possible to find a solution of the first equation of (5') only if

$$\int \left[\int \phi_2(y) \sum_{i=1}^m \xi^{(i)}(y) \varphi_1^{(i)}(x) dy \right] \varphi_1^{(l)}(x) dx = 0,$$

$l=1, 2, \dots, m.$

(1) Cf. Schmidt, *Mathematische Annalen*, vol. 63 (1907), p. 449.

But since the $\varphi_1^{(l)}(x)$ are normed and orthogonal functions we must have

$$(9) \quad \int \phi_2(y) \xi^{(l)}(y) dy = 0.$$

The solution of the first equation of (5') is then

$$\begin{aligned} \phi_1(x) = & \int \phi_2(y) \sum_{i=1}^m \xi^{(i)}(y) \varphi_1^{(i)}(x) dy + \int \left[\int \phi_2(z) \sum_{i=1}^m \xi^{(i)}(z) \varphi_1^{(i)}(y) dz \right] \\ & \times L(y, x; k_{11}) dy + \sum_{i=1}^m a_i \phi_1^{(i)}(x), \end{aligned}$$

where $L(y, x; k_{11})$ is the pseudo-reciprocal kernel of $k_{11}(x, y)$ and the $\phi_1^{(i)}(x)$ are the solutions of (6).

Taking (9) into consideration, this value of $\phi_1(x)$ becomes

$$\phi_1(x) = \sum_{i=1}^m a_i \varphi_1^{(i)}(x).$$

Substituting then, this value of $\phi_1(x)$ in the second equation of (5') we have

$$\phi_2(x) - \int \sum_{i=1}^m a_i \phi_1^{(i)}(y) k_{12}(y, x) dy - \int \phi_2(y) k_{22}(y, x) dy = 0.$$

Since the Fredholm determinant of $k_{22}(x, y)$ is evidently zero, the condition that this equation have a solution is that the relation

$$(10) \quad \int \left[\int \sum_{i=1}^m a_i \phi_1^{(i)}(y) k_{12}(y, x) dy \right] \varphi_2^{(l)}(x) dx = 0$$

$l = 1, 2, \dots, m$

shall be satisfied, the $\varphi_2^{(l)}(x)$ being the non-zero solutions of (8).

But

$$\varphi_2^{(l)}(x) = \int \phi_1^{(l)}(y) k_{12}(y, x) dy \quad l = 1, 2, \dots, m.$$

Multiplying (10) through by a_l and summing as to l we get

$$\int \left[\sum_{i=1}^m a_i \varphi_2^{(i)}(x) \right] \left[\sum_{l=1}^m a_l \varphi_2^{(l)}(x) \right] dx = \int \left[\sum_{i=1}^m a_i \varphi_2^{(i)}(x) \right]^2 dx = 0$$

and hence $\sum_{i=1}^m a_i \varphi_2^{(i)}(x) = 0$, i.e. the $\varphi_2^{(i)}(x)$ being linearly independent, we must have $a_i = 0$ and consequently $\phi_1(x) = 0$.

The second equation of (5'), then, becomes

$$\phi_2(x) - \int \sum_{i=1}^m [\phi_2(y) \xi^{(i)}(y) dy] \xi^{(i)}(x) = 0$$

and hence by (9), we must have $\phi_2(x) = 0$.

If then the $\phi_2^{(i)}(x)$ are linearly independent, the only possible solution of the system (5') is $\phi_1(x) = \phi_2(x) = 0$ which is contrary to the hypothesis. Hence the condition (b) must be fulfilled, i.e. there exists a set of constants α_i not all zero such that

$$\sum_{i=1}^m \alpha_i \phi_2^{(i)}(x) = \sum_{i=1}^m \alpha_i \int \phi_1^{(i)}(y) k_{12}(y, x) dy = \int \sum_{i=1}^m \alpha_i \phi_1^{(i)}(y) k_{12}(y, x) dy = 0.$$

But since $\sum_{i=1}^m \alpha_i \phi_1^{(i)}(x)$ is a solution of (6) we have shown that in order that (1) be linearly dependent it is necessary that the equations (4) have a common solution.

Finally with the aid of the theorem just proved, we shall prove the following

Theorem II. A necessary and sufficient condition that (1) be linearly dependent is that there exists a function $\phi(x)$ such that $\int \phi(x) f_1(x) dx = 0$ for every $\phi_1(x)$ and $\phi_2(x)$.

Suppose (1) is linearly dependent. Then by Theorem I there exists a function $\phi_1(x)$ which satisfies the relations (4). Multiplying (1) through by this $\phi_1(x)$ and integrating we get

$$\begin{aligned} \int \phi_1(x) f_1(x) dx &= \int \phi_1(x) \varphi_1(x) dx - \iint \phi_1(x) k_{11}(x, y) \varphi_1(y) dy dx \\ &\quad - \iint \phi_1(x) k_{12}(x, y) \varphi_2(y) dy dx \\ &= \int \left[\phi_1(x) - \int \phi_1(y) k_{11}(y, x) dy \right] \varphi_1(x) dx \\ &\quad - \int \left[\int \phi_1(y) k_{12}(y, x) dy \right] \varphi_2(x) dx. \end{aligned}$$

But by (4) the right hand member of this last identity is zero. Hence the left hand member must also be zero, i.e. we must have

$$\int \phi_1(x) f_1(x) dx = 0$$

and we have shown that the condition of our theorem is necessary.

To prove that the condition is also sufficient let us assume the

existence of a function $\phi_1(x)$ such that $\int \phi_1(x) f_1(x) dx = 0$ for every $\varphi_1(x)$ and $\varphi_2(x)$. Then if we multiply (1) through by this $\phi_1(x)$ and integrate, we shall have

$$\int \left[\phi_1(x) - \int \phi_1(y) k_{11}(y, x) dy \right] \varphi_1(x) dx - \int \left[\int \phi_1(y) k_{12}(y, x) dy \right] \varphi_2(x) dx = 0$$

for every $\varphi_1(x)$ and $\varphi_2(x)$.

For

$$\varphi_1(x) = \phi_1(x) - \int \phi_1(y) k_{11}(y, x) dy \quad \text{and} \quad \varphi_2(x) = - \int \phi_1(y) k_{12}(y, x) dy,$$

this identity becomes

$$\int \left[\phi_1(x) - \int \phi_1(y) k_{11}(y, x) dy \right]^2 dx + \int \left[\int \phi_1(y) k_{12}(y, x) dy \right]^2 dx = 0,$$

i.e. the function $\phi_1(x)$ must satisfy the relations

$$\phi_1(x) - \int \phi_1(y) k_{11}(y, x) dy = 0,$$

$$\int \phi_1(y) k_{12}(y, x) dy = 0.$$

But by Theorem I this condition is sufficient for linear dependence in (1). It will be noticed that the condition contained in the theorem just proved is analogous to the condition for linear dependence which one finds in connection with the theory of algebraic equations.

PART II.

Consider now the system of linear integral expressions

$$(1) \quad \varphi_i(x) - \sum_{j=1}^n \int k_{ij}(x, y) \varphi_j(y) dy = f_i(x) \\ i = 1, 2, 3, \dots, m.$$

This system is said to be linearly dependent if there exists a non-zero solution of the system,

$$(2) \quad \varphi_i(x) - \sum_{j=1}^n \int k_{ij}(x, y) \varphi_j(y) dy = 0 \\ i = 1, 2, 3, \dots, m \\ \varphi_h(x) - \sum_{j=1}^n \int k_{hj}(x, y) \varphi_j(y) dy = 0 \\ h = m+1, m+2, \dots, n.$$

whatever be the functions $k_{h,j}(x, y)$ in the last $n-m$ equations.

It is possible to obtain a condition for linear dependence in (1) in terms of the Fredholm determinants and their minors for the system, but the results are rather complicated. However, the theorems of Part I, § 3 generalize themselves very readily and the demonstrations are closely analogous, so we shall confine ourselves to these in this section.

We have then relative to the system (1)

Theorem I. A necessary and sufficient condition that (1) be linearly dependent is that there exists a set of m functions $\phi_1(x)$, $\phi_2(x)$, $\dots$, $\phi_m(x)$, not all zero which satisfy the relations

$$\begin{aligned} \phi_i(x) - \sum_{l=1}^m \int \phi_l(y) k_{li}(y, x) dy &= 0 \\ (3) \qquad \qquad \qquad i &= 1, 2, 3, \dots, m. \\ \sum_{l=1}^m \int \phi_l(y) k_{lh}(y, x) dy &= 0 \\ h &= m+1, m+2, \dots, n. \end{aligned}$$

Suppose there exists a set of functions $\phi_1^{(0)}(x)$, $\phi_2^{(0)}(x)$, $\dots$, $\phi_m^{(0)}(x)$, not all of which are identically zero which satisfy the relations (3). Then these $\phi_i^{(0)}(x)$ together with $\phi_h^{(0)}(x) = 0$ ($h = m+1, m+2, \dots, n$) would obviously constitute a non-zero solution of the system.

$$\begin{aligned} \phi_i(x) - \sum_{j=1}^n \int \phi_j(y) k_{ji}(y, x) dy &= 0 \\ (4) \qquad \qquad \qquad i &= 1, 2, \dots, m, \\ \phi_h(x) - \sum_{j=1}^n \int \phi_j(y) k_{jh}(y, x) dy &= 0 \\ h &= m+1, m+2, \dots, n, \end{aligned}$$

which is the homogeneous adjoint system of (2).

But the existence of a non-zero solution of (4) implies the existence of a non-zero solution of (2). The expressions (1) are then linearly dependent and we have shown that the condition of our theorem is sufficient.

To show that the condition is also necessary, i.e. there exists a solution of the equations (3), let us assume that for every $k_{h,j}(x, y)$ ($h = m+1, m+2, \dots, n; j = 1, 2, \dots, n$) there exists a set of functions $\varphi_1(x)$, $\varphi_2(x)$, $\dots$, $\varphi_n(x)$, not all of which are zero, which satisfies the system (2). Then for every $k_{h,j}(x, y)$ there will also exist a set of functions $\psi_1(x)$, $\psi_2(x)$, $\dots$, $\psi_n(x)$, not all zero, which satisfies the system (4).

First let us show that we must have

$$F_{\circ}[k_{il}]=0 \quad l, i=1, 2, \dots, m,$$

where $F_{\circ}[k_{il}]$ denotes the Fredholm determinant of the system of equations whose functions are the $k_{il}(x, y)$ ($l, i=1, 2, \dots, m$).

Since the $k_{hj}(x, y)$ are arbitrary let us choose $k_{hl}(x, y)=0$ ($h=m+1, m+2, \dots, n; i=1, 2, m$). Then from the first m equations of (4) we have

$$(5) \quad \phi_i(x) - \sum_{l=1}^m \int \phi_l(y) k_{il}(y, x) dy = 0 \quad i=1, 2, \dots, m.$$

There are now two cases to be considered:

$$(a) \quad F_{\circ}[k_{il}] \neq 0, \quad l, i=1, 2, \dots, m.$$

$$(b) \quad F_{\circ}[k_{il}]=0.$$

If (a) is true the only possible solutions of (5) would be $\phi_1(x)=\phi_2(x)=\dots=\phi_m(x)=0$ and the last $n-m$ equations of (4) would give

$$\phi_h(x) - \sum_{p=m+1}^n \int \phi_p(y) k_{ph}(y, x) dy = 0 \quad h=m+1, m+2, \dots, n.$$

Then in order to have a non-zero solution of (4) it would be necessary to have $F_{\circ}[k_{ph}]=0$ for all $k_{ph}(x, y)$, ($p, h=m+1, m+2, \dots, n$) which is impossible. We then must have $F_{\circ}[k_{il}]=0$ ($l, i=1, 2, \dots, m$), i.e. (b) must be true, and there exist solutions of the system (5) and also solutions of the system

$$(6) \quad \phi_i(x) - \sum_{l=1}^m \int k_{il}(x, y) \phi_l(y) dy = 0 \quad i=1, 2, \dots, m.$$

Now let us show that some solution of the first m of the equations (3) satisfies also the last $n-m$.

Suppose the r normed and orthogonal sets of functions $\varphi_1^{(q)}(x), \varphi_2^{(q)}(x), \dots, \varphi_m^{(q)}(x)$, ($q=1, 2, \dots, r$) constitute a complete linearly independent set of solutions of the system (6). Suppose also the r sets of functions $\phi_1^{(q)}(x), \phi_2^{(q)}(x), \dots, \phi_m^{(q)}(x)$, ($q=1, 2, \dots, r$) constitute a complete linearly independent set of solutions of the system (5). Let us substitute these $\phi_i^{(q)}(x)$ for $\phi_i(x)$ in the last $n-m$ equations of (4) and place

$$\varphi_h^{(q)}(x) = \sum_{i=1}^m \int \phi_i^{(q)}(y) k_{ih}(y, x) dy \quad \begin{matrix} h=m+1, m+2, \dots, n, \\ q=1, 2, \dots, r. \end{matrix}$$

Suppose, if possible, that these r sets of functions $\varphi_h^{(q)}(x)$, ($q=1, 2, \dots, r$), are linearly independent. Then there exist r equivalent sets of functions $\xi_{m+1}^{(q)}(x)$, $\xi_{m+2}^{(q)}(x)$, $\dots$, $\xi_n^{(q)}(x)$ such that

$$\sum_{p=m+1}^n \int \xi_p^{(q)}(x) \xi_p^{(s)}(x) dx = \begin{cases} 0 & \text{for } q \neq s \\ 1 & \text{for } q = s \end{cases}$$

and such that

$$\xi_h^{(q)}(x) = \sum_{q=1}^r c_q \varphi_h^{(q)}(x) \quad h=m+1, m+2, \dots, n.$$

Conversely, the $\varphi_h^{(q)}(x)$ are linear combinations of the $\xi_h^{(q)}(x)$.

Let us then choose $k_{hi}(x, y)$ ($i=1, 2, \dots, m$) such that

$$k_{hi}(x, y) = \sum_{q=1}^r \xi_h^{(q)}(x) \varphi_i^{(q)}(y) \quad \begin{aligned} h &= m+1, m+2, \dots, n. \\ i &= 1, 2, \dots, m, \end{aligned}$$

and $k_{hp}(x, y)$ ($h, p=m+1, m+2, \dots, n$) such that

$$k_{hp}(x, y) = \sum_{q=1}^r \xi_h^{(q)}(x) \xi_p^{(q)}(y) \quad h, p=m+1, m+2, \dots, n.$$

The Fredholm determinant of the system $k_{hp}(x, y)$ ($h, p=m+1, m+2, \dots, n$) will then evidently be zero, and the functions $\varphi_h^{(q)}(x)$ will constitute a complete linearly independent set of solution of the system

$$(7) \quad \varphi_h(x) - \sum_{p=m+1}^n \int k_{hp}(x, y) \varphi_p(y) dy = 0 \quad h=m+1, m+2, \dots, n.$$

If then the system

$$(4') \quad \begin{aligned} \varphi_i(x) - \sum_{l=1}^m \int \varphi_l(y) k_{li}(y, x) dy - \sum_{h=m+1}^n \int \varphi_h(y) \sum_{q=1}^r \xi_h^{(q)}(y) \varphi_i^{(q)}(x) dy &= 0 \\ i &= 1, 2, \dots, m \\ \varphi_p(x) - \sum_{l=1}^m \int \varphi_l(y) k_{lp}(y, x) dy - \sum_{h=m+1}^n \int \varphi_h(y) \sum_{q=1}^r \xi_h^{(q)}(y) \xi_p^{(q)}(x) dy &= 0 \\ p &= m+1, m+2, \dots, n, \end{aligned}$$

is to have a solution not identically zero, it must be possible to solve the first m equations for $\varphi_i(x)$ ($i=1, 2, \dots, m$) in terms of the $\varphi_h(x)$

($h=m+1, m+2, \dots, n$) and then substitute these values of $\phi_i(x)$ in the last ($n-m$) equations and solve for $\phi_h(x)$.

Since $F_0[k_u]=0$ $l, i=1, 2, \dots, m,$

it is possible to find a solution of the first m equations of (4') only if

$$\int_{i=1}^m \left[\sum_{h=m+1}^n \int \phi_h(y) \sum_{q=1}^r \xi_h^{(q)}(y) \varphi_i^{(q)}(x) dy \right] \varphi_i^{(s)}(x) dx = 0$$

$s=1, 2, \dots, r.$

But since the $\varphi_i^{(q)}(x)$ are normed and orthogonal sets of functions we must have

$$(8) \quad \sum_{h=m+1}^n \int \phi_h(y) \sum_{q=1}^r \xi_h^{(q)}(y) dy = 0.$$

The solution of the first m equations of (4') is then

$$\phi_i(x) = \sum_{q=1}^r a_q \varphi_i^{(q)}(x) \quad i=1, 2, \dots, m,$$

where the $\varphi_i^{(q)}(x)$ are the solutions of the system (5).

Substituting these values of $\phi_i(x)$ in the last ($n-m$) equations of (4') we get

$$(9) \quad \phi_h(x) - \sum_{p=m+1}^n \int \phi_p(y) k_{hp}(y, x) dy = \sum_{i=1}^m \int \sum_{q=1}^r a_q \varphi_i^{(q)}(y) k_{ih}(y, x) dy$$

$h=m+1, m+2, \dots, n.$

Since the Fredholm determinant of $k_{hp}(x, y)$ ($h, p=m+1, m+2, \dots, n$) is obviously zero and since the $\varphi_h^{(q)}(x)$ are the solutions of the homogeneous system adjoint to the system (9), the condition for a solution of (9) is

$$(10) \quad \int_{h=m+1}^n \sum_{q=1}^r \left[\sum_{q=1}^r a_q \varphi_h^{(q)}(x) \right] \varphi_h^{(s)}(x) dx = 0, \quad s=1, 2, \dots, r.$$

Multiplying (10) through by a_s and summing as to s we get

$$\begin{aligned} \sum_{s=1}^r a_s \int_{h=m+1}^n \sum_{q=1}^r \left[\sum_{q=1}^r a_q \varphi_h^{(q)}(x) \right] \varphi_h^{(s)}(x) dx &= \int_{h=m+1}^n \sum_{q=1}^r \left[\sum_{q=1}^r a_q \varphi_h^{(q)}(x) \right]^2 dx \\ &= \sum_{h=m+1}^n \int \left[\sum_{q=1}^r a_q \varphi_h^{(q)}(x) \right]^2 dx = 0 \end{aligned}$$

and hence

$$\sum_{q=1}^r a_q \varphi_h^{(q)}(x) = 0, \quad h=m+1, m+2, \dots, n.$$

i.e. the $\varphi_h^{(q)}(x)$ being linearly independent sets of functions, we must have $\alpha_q=0$ ($q=1, 2, \dots, r$) and consequently $\phi_i(x)=0$ ($i=1, 2, \dots, m$).

The last $(n-m)$ equations of (4') then become

$$\phi_p(x) - \sum_{h=m+1}^n \int \phi_h(y) \sum_{q=1}^r \xi_h^{(q)}(y) \xi_p^{(q)}(x) dy = 0$$

and hence by (8) we must have $\phi_p(x)=0$ ($p=m+1, m+2, \dots, n$).

If then, the sets of functions $\varphi_h^{(q)}(x)$ are linearly independent the only possible solution of the system (4') is $\phi_1(x) = \dots = \phi_n(x) = 0$ which is contrary to the hypothesis. Hence the $\varphi_h^{(p)}(x)$ must be linearly dependent; i.e. there exists a set of constants α such that

$$\begin{aligned} \sum_{q=1}^r \alpha_q \varphi_h^{(q)}(x) &= \sum_{q=1}^r \alpha_q \sum_{i=1}^m \int \phi_i^{(q)}(x) k_{ih}(y, x) dy \\ &= \sum_{i=1}^m \int \sum_{q=1}^r \alpha_q \phi_i^{(q)}(x) k_{ih}(y, x) dy = 0. \end{aligned}$$

But since $\sum_{q=1}^r \alpha_q \phi_i^{(q)}(x)$ ($i=1, 2, \dots, m$) is a solution of (5) we have shown that for the expressions (1) to be linearly dependent it is necessary that there exists a non-zero solution of the equations (3).

Relative to the system (1) we have also the following

Theorem II. A necessary and sufficient condition that the linear integral expressions (1) be linearly dependent is that there exists a set of functions $\phi_1(x), \phi_2(x), \dots, \phi_m(x)$ not all zero such that $\sum_{i=1}^m \int \phi_i(x) f_i(x) dx = 0$ for every $\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)$.

This Theorem is easily proved by the method adopted in proving its analogue in the case of a single linear integral expression, i.e. Theorem II of Part I, § 3. For, suppose the expressions (1) are linearly dependent. Then by Theorem I there exists a set of functions $\phi_1(x), \phi_2(x), \dots, \phi_m(x)$, which satisfy the relations (3). If we multiply the expressions (1) through by these $\phi_i(x)$, integrate, and add, we get

$$\begin{aligned} \sum_{i=1}^m \int \left(\phi_i(x) - \sum_{h=m+1}^n \int \phi_h(y) k_{ih}(y, x) dy \right) \varphi_i(x) dx &- \sum_{h=m+1}^n \int \left[\sum_{i=1}^m \int \phi_i(y) \right. \\ &\times k_{ih}(y, x) dy \left. \right] \varphi_h(x) dx = \sum_{i=1}^m \int \phi_i(x) f_i(x) dx. \end{aligned}$$

But the left hand member of this identity is zero, hence the right

hand member must also be zero, i.e. we must have $\sum_{i=1}^m \int \phi_i(x) f_i(x) dx = 0$

and we have shown that our condition is necessary.

On the other hand, the condition is sufficient. For if there exists a set of functions $\phi_1(x), \phi_2(x), \dots, \phi_m(x)$ such that $\sum_{i=1}^m \int \phi_i(x) f_i(x) dx = 0$, and if we multiply the expressions (1) through by these $\phi_i(x)$, integrate and add we shall have

$$\begin{aligned} \sum_{i=1}^m \int \left(\phi_i(x) - \sum_{l=1}^m \int \phi_l(y) k_{il}(y, x) dy \right) \varphi_i(x) dx \\ - \sum_{i=1}^m \int \sum_{h=m+1}^n \left(\int \phi_i(x) k_{ih}(y, x) dy \right) \varphi_h(x) dx = 0 \end{aligned}$$

for every $\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)$. Then for

$$\begin{aligned} \varphi_i(x) = \phi_i(x) - \sum_{l=1}^m \int \phi_l(y) k_{il}(y, x) dy \\ i = 1, 2, 3, \dots, m. \end{aligned}$$

$$\begin{aligned} \varphi_h(x) = - \sum_{i=1}^m \int \phi_i(x) k_{ih}(y, x) dy \\ h = m+1, m+2, \dots, n, \end{aligned}$$

the above identity becomes

$$\begin{aligned} \sum_{i=1}^m \int \left[\phi_i(x) - \sum_{l=1}^m \int \phi_l(y) k_{il}(y, x) dy \right]^2 dx \\ + \sum_{h=m+1}^n \int \left[\sum_{i=1}^m \int \phi_i(x) k_{ih}(y, x) dy \right]^2 dy = 0, \end{aligned}$$

i.e. we must have

$$\begin{aligned} \phi_i(x) - \sum_{l=1}^m \int \phi_l(y) k_{il}(y, x) dy = 0 \\ i = 1, 2, 3, \dots, m, \\ \sum_{i=1}^m \int \phi_i(y) k_{ih}(y, x) dy = 0 \\ h = m+1, m+2, \dots, n. \end{aligned}$$

But by Theorem I, this is a sufficient condition for linear dependence in (1).

We have thus shown that in the definition of linear dependence for a system of linear integral expressions:

$$\varphi_i(x) - \sum_{j=1}^n \int k_{ij}(x, y) \varphi_j(x) dx = f_i(x) \quad i=1, 2, 3, \dots, m,$$

we can proceed in three equivalent ways analogous to the situation for algebraic equations viz:

(a) for every $k_{hj}(x, y)$, ($h=m+1, m+2, \dots, n$; $j=1, 2, 3, \dots, n$) the Fredholm determinant of the system

$$\begin{aligned} \varphi_i(x) - \sum_{j=1}^n \int k_{ij}(x, y) \varphi_j(y) dy &= 0 \\ i &= 1, 2, \dots, m, \\ \varphi_h(x) - \sum_{j=1}^n \int k_{hj}(x, y) \varphi_j(y) dy &= 0 \\ h &= m+1, m+2, \dots, n, \end{aligned}$$

is identically zero;

(b) there exists at least one solution of the adjoint system,

$$\begin{aligned} \psi_i(x) - \sum_{l=1}^m \int \psi_l(y) k_{li}(y, x) dy &= 0 \\ i &= 1, 2, \dots, m, \\ \sum_{l=1}^m \int \psi_l(y) k_{ln}(y, x) dy &= 0 \\ h &= m+1, m+2, \dots, n, \end{aligned}$$

and

(c) there exists a set of functions $\psi_i(x)$, ($i=1, 2, 3, \dots, m$) such that

$$\sum_{i=1}^m \int f_i(x) \psi_i(x) dx = 0$$

for every function $\psi_j(x)$ ($j=1, 2, 3, \dots, n$).

Finally we might note that the above theory is easily extensible in the sense of E. H. Moore's general integral equation theory⁽¹⁾.

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